UDC 539.31 : 539.375

TORSION AND EXTENSION OF A CYLINDER WITH AN EXTERNAL ANNULAR SLIT

PMM Vol. 37, №2, 1973, pp. 316-325

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(Received June 16, 1972)

The state of stress is determined in the neighborhood of an annular slit on the surface of an infinite solid cylinder. By using the results of [1], stress intensity coefficients are found in the case of pure torsion and extension along the cylinder axis.

The pertinence of a state of stress analysis for a notched circular sample under torsion and extension is governed by the fact that the sample shape turns out to be important for standard fracture tests.

1. Let a solid circular cylinder of unit radius with annular notch with inner diameter $2a$ be twisted by moments M (Fig. 1). The cylinder is referred to a cylindrical system of (r, θ, z) coordinates with center in the plane of the slit. It is assumed that the side surface of the cylinder and the surface of the slit are stress-free. The shear stresses in the elementary solution of the problem of torsion of a shaft of unit radius are the following:

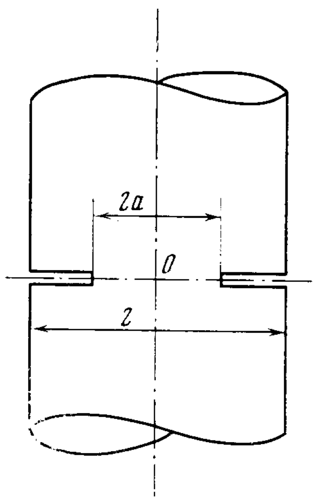

Fig. 1

$$\tau_{\theta z}^{(0)} = \frac{2M}{\pi} r, \qquad \tau_{r\theta}^{(0)} = 0 \tag{1.1}$$

In order for the surface of the slit to be stress-free, it is necessary to consider an additional state of stress which is independent of the angular coordinate θ and is characterized by a single nonzero component of the displacement $u_\theta = u(r, z)$ which satisfies the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1.2}$$

The shear stresses are defined by the formulas

$$\tau_{\theta z}^{(1)} = \mu \frac{\partial u}{\partial z}, \qquad \tau_{r\theta}^{(1)} = \mu \left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) \tag{1.3}$$

Let us consider a semi-infinite cylinder ($z \geqslant 0$) and let us represent the solution of (1.2) as

$$u(r, z) = 2 \sum_{n=1}^{\infty} \frac{B_n J_1(\lambda_n z)}{\lambda_n} \exp(-\lambda_n r) \tag{1.4}$$

Here $J_1(\lambda_n z)$ is a Bessel's function of the first kind and λ_n are roots of the equation

$$\lambda_n J_1'(\lambda_n) - J_1(\lambda_n) = -\lambda_n J_2(\lambda_n) = 0$$

The shear stresses on the side surface are zero, and are determined as follows on the endface in conformity with (1.3), (1.4):

$$\tau_{\theta z}^{(1)} = \mu \frac{\partial u}{\partial z} = -2\mu \sum_{n=1}^{\infty} B_n J_1(\lambda_n r), \quad z = 0 \tag{1.5}$$

The boundary conditions in the $z = 0$ plane are

$$\tau_{\theta z}^{(1)}(r, 0) = -\tau_{\theta z}^{(0)}(r, 0), \qquad a < r \leqslant 1$$

$$u(r, 0) = 0, \qquad 0 \leqslant r < a \tag{1.6}$$

Satisfying these boundary conditions and using the relationships (1.4), (1.5), we obtain the dual series equations

$$\sum_{n=1}^{\infty} \frac{B_n}{\lambda_n} J_1(\lambda_n r) = 0, \qquad 0 \leqslant r < a \tag{1.7}$$

$$\sum_{n=1}^{\infty} B_n J_1(\lambda_n r) = \frac{M}{\pi\mu} r, \qquad a < r \leqslant 1 \tag{1.8}$$

Following [1], let us set

$$-\frac{M}{\pi\mu} r + \sum_{n=1}^{\infty} B_n J_1(\lambda_n r) = -\frac{\partial}{\partial r} \int_r^a \frac{g(t)\, dt}{\sqrt{t^2 - r^2}}, \qquad r < a \tag{1.9}$$

Then on the basis of (1.8), (1.9), we find

$$B_n = 2J_1^{-2}(\lambda_n) \int_0^a g(u) \sin(\lambda_n u)\, du \tag{1.10}$$

$$-\frac{M}{\pi\mu} = 8 \int_0^a u g(u)\, du \tag{1.11}$$

The function $g(u)$ satisfies a Fredholm integral equation of the second kind with the symmetric kernel [1]

$$g(t) = \int_0^a g(u) K(u, t)\, dt + \frac{16}{\pi} t \int_0^a u g(u)\, du \tag{1.12}$$

$$K(u, t) = \frac{4}{\pi^2} \int_0^{\infty} \frac{K_2(y)}{I_2(y)} [8utI_2(y) - \operatorname{sh}(ty)\operatorname{sh}(uy)]\, dy \tag{1.13}$$

Here $I_n(y)$, $K_n(y)$ are the modified Bessel's functions of the first and second kind, respectively. Using the power series expansions for $\operatorname{sh}(ty)$, $\operatorname{sh}(uy)$ and $I_2(y)$, let us represent the kernel in (1.12) as

$$K(u, t) = -\sum_{k=0}^{\infty} t^{2k+1} b_{2k+1}(u) \tag{1.14}$$

$$b_1(u) = \sum_{n=2}^{\infty} \alpha_n \left[\frac{u^{2n-1}}{(2n-1)!} - \frac{u}{2^{2n-3}(n-1)!\,(n+1)!} \right]$$

$$b_{2k+1}(u) = \frac{1}{(2k+1)!} \sum_{n=2}^{\infty} \frac{\alpha_{n+k-1} u^{2n-3}}{(2n-3)!} \qquad (k = 1, 2, \ldots) \tag{1.15}$$

$$\alpha_n = \frac{4}{\pi^2}\int_0^\infty \frac{K_2(y)}{I_2(y)} y^{2n} dy \qquad (n = 2, 3, \ldots) \tag{1.16}$$

Taking account of the expansion (1.14), the solution of the integral equation (1.12) will be sought in the form

$$g(t) = C(a)\sum_{m=0}^\infty P_{2m+1} t^{2m+1} \tag{1.17}$$

The constants P_{2m+1} in (1.17) are determined from the solution of the infinite system of algebraic equations

$$P_{2m+1} = -\sum_{k=0}^\infty P_{2k+1} C_{2k+1,\,2m+1} + \delta_m^\circ \qquad (m = 0, 1, 2, \ldots) \tag{1.18}$$

Here

$$C_{2k+1,\,1} = \sum_{n=2}^\infty \alpha_n \left[\frac{a^{2k+2n+1}}{(2k+2n+1)(2n-1)!} - \frac{a^{2k+3}}{2^{2n-3}(2k+3)(n-1)!\,(n+1)!}\right]$$

$$C_{2k+1,\,2m+1} = \int_0^a u^{2k+1} b_{2m+1}(u)\, du = \frac{1}{(2m+1)!}\sum_{n=2}^\infty \frac{\alpha_{n+m-1}\, a^{2k+2n-1}}{(2k+2n-1)(2n-3)!} \tag{1.19}$$

$$k = 0, 1, 2, \ldots;\ m = 1, 2, 3, \ldots, \qquad \delta_m^\circ = \begin{cases} 1, & m = 0 \\ 0, & m = 1, 2, \ldots \end{cases}$$

The constant $C(a)$ is determined from the equality (1.11) and the expansion (1.17)

$$C(a)\sum_{m=0}^\infty P_{2m+1}\frac{a^{2m+3}}{2m+3} = -\frac{M}{8\pi\mu} \tag{1.20}$$

The system (1.18) is quasi-regular. In order to show this, let us first prove that

$$\lim_{m\to\infty} b_{2m+1}(u) = 0, \qquad 0 \leqslant u < 1 \tag{1.21}$$

We substitute the expression for the coefficients α_n in the series defining $b_{2m+1}(u)$, and we change the order of summation and integration. We then obtain

$$b_{2m+1}(u) = \frac{4}{\pi^2(2m+1)!}\int_0^\infty \frac{K_2(y)}{I_2(y)} y^{2m+1}\,\mathrm{sh}(uy)\, dy \qquad (m = 1, 2, \ldots)$$

Using the asymptotic expansions for the modified Bessel's functions, we obtain for large m

$$b_{2m+1}(u) \sim \frac{2}{\pi(2-u)^{2m+1}}\left[1 + \frac{15(2-u)}{4(2m+1)} + \frac{225(2-u)^2}{32(2m+1)2m} + \ldots\right] - \frac{2}{\pi(2+u)^{2m+1}}\left[1 + \frac{15(2+u)}{4(2m+1)} + \frac{225(2+u)^2}{32(2m+1)2m} + \ldots\right] \tag{1.22}$$

Hence, (1.21) follows. Let us note that the values of α_n for large n can be determined from the asymptotic formula

$$\alpha_n \sim \frac{(2n)!}{2^{2n-1}\pi}\left[1 + \frac{15}{4}\frac{2}{2n} + \frac{225}{32}\frac{2^2}{(2n-1)(2n)} + \ldots\right] \tag{1.23}$$

Values of α_n according to (1.16) (first row) and (1.23) are presented in Table 1.

Table 1

2	3	4	5	6	7	8
$117\cdot10^{-1}$ $80\cdot10^{-1}$	$297\cdot10^{-1}$ $274\cdot10^{-1}$	276 267	$498\cdot10^{1}$ $489\cdot10^{1}$	$142\cdot10^{3}$ $141\cdot10^{3}$	$587\cdot10^{4}$ $584\cdot10^{4}$	$326\cdot10^{6}$ $327\cdot10^{6}$

9	10	11	12	13	14	15
$235\cdot10^{8}$ $237\cdot10^{8}$	$212\cdot10^{10}$ $216\cdot10^{10}$	$235\cdot10^{12}$ $241\cdot10^{12}$	$312\cdot10^{14}$ $323\cdot10^{14}$	$490\cdot10^{16}$ $511\cdot10^{16}$	$894\cdot10^{18}$ $946\cdot10^{18}$	$189\cdot10^{21}$ $202\cdot10^{21}$

A comparison shows that the exact values of α_n for $n \geqslant 5$ differ slightly from the asymptotic values.

Estimating the coefficients of the infinite system (1.18) we find

$$|C_{2k+1,\,2m+1}| \leqslant |b_{2m+1}(a_1)|\,\frac{a^{2k+2}}{2k+2}, \qquad 0 \leqslant a_1 \leqslant a < 1$$

Consequently

$$S_{2m+1} = \sum_{k=1}^{\infty} |C_{2k+1,\,2m+1}| \leqslant \frac{1}{2}\,|b_{2m+1}(a_1)|\,|\ln(1-a^2)+a^2| \tag{1.24}$$

We obtain from (1.23), (1.24) that starting with some number $m = m'$ the following inequality holds:

$$S_{2m+1} < 1, \qquad m \geqslant m'$$

Thus, the system (1.18) is quasi-regular for $0 \leqslant a < 1$.

Let us use the relationships (1.5), (1.9) to determine the shear stresses on a continuation of the slit

$$\tau_{\theta z}^{(1)}(r,\,0) = -\frac{2}{\pi}Mr + 2\mu\frac{\partial}{\partial r}\int_r^a \frac{g(t)\,dt}{\sqrt{t^2-r^2}}, \qquad r < a \tag{1.25}$$

If the expressions (1.17), (1.20) are substituted into (1.25), then it can be shown that the shear stress $\tau_{\theta z}^{(1)}$ on the continuation of the slit will have a singularity of the following form:

$$\tau_{\theta z}^{(1)}(r,\,0) = -\frac{2\mu C(a)}{\sqrt{a^2-r^2}}\sum_{m=0}^{\infty} P_{2m+1}r^{2m+1} + \ldots, \qquad r < a \tag{1.26}$$

Taking account of (1.26), we find the stress intensity coefficient at the slit vertex [2]

$$K_{\rm III} = \lim_{r\to a}\,[\sqrt{2\pi(a-r)}\,\tau_{\theta z}(r,\,0)] = -2\mu\sqrt{\frac{\pi}{a}}\,g(a), \qquad r < a \tag{1.27}$$

Using the expansion (1.17), we can write

$$K_{\rm III} = \frac{M}{4\sqrt{\pi a}}\left(\sum_{m=0}^{\infty} P_{2m+1}a^{2m+1}\right)\left(\sum_{m=0}^{\infty} P_{2m+1}\frac{a^{2m+3}}{2m+3}\right)^{-1} \tag{1.28}$$

The solid line in Fig. 2 represents the dependence of the quantity

$$M^* = {}^3/_4 M\pi^{-1/2}R^{-5/2}K_{IIIc}^{-1}$$

on the dimensionless radius ($\alpha = a/R$) of the vertex of the crack (R is the cylinder radius introduced instead of the unit radius. It is seen that for small a (a deep annular crack), $P_1 \sim 1$, and $P_{2k+1} \sim 0$ $(k = 1,\,2,\,\ldots)$. Then it follows from (1.28)

$$K_{\rm III} \sim {}^3/_4 M\pi^{-1/2}a^{-3/2} \qquad (1.29)$$

This result (the dashed line in Fig. 2), follows from the known Neuber [3] solution of the problem of torsion of a body of revolution containing an external groove. Conversely, for shallow grooves on the surface of a cylinder, a half-plane with a slit emerging on its boundary can be considered under the conditions of antiplane strain (Fig. 3). In this case the boundary conditions are the following:

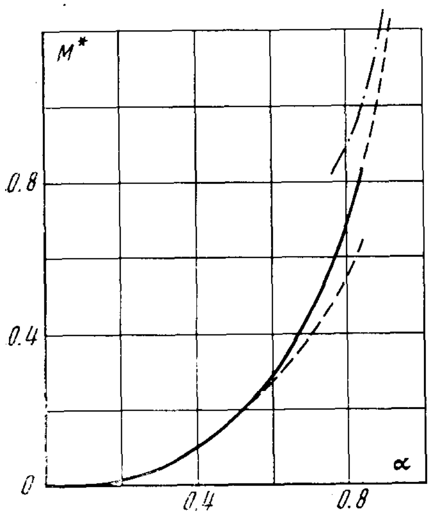

Fig. 2

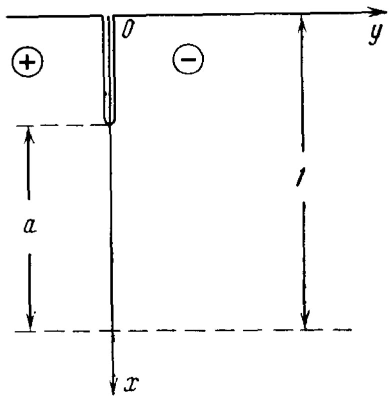

Fig. 3

$$\tau_{zx} = 0, \quad x = 0; \quad u = 0, \quad y = 0, \quad 1 - a \leqslant x$$
$$\tau_{zy} = -\frac{2M}{\pi}(1 - x), \qquad y = 0, \quad 0 \leqslant x \leqslant 1 - a \qquad (1.30)$$

The last condition corresponds to the selection of the stresses (with opposite sign) originating at the location of the shallow groove in the torsion of a solid cylinder of unit radius. In this case we have

$$\Delta u = 0, \quad \tau_{zy} = \mu \frac{\partial u}{\partial y}, \quad \tau_{zx} = \mu \frac{\partial u}{\partial x}$$

Let us represent the displacement and stress components as follows:

$$u(x, y) = \int_0^\infty A(\lambda) e^{-\lambda y} \cos \lambda x d\lambda$$

$$\tau_{zx} = -\mu \int_0^\infty \lambda A(\lambda) e^{-\lambda y} \sin \lambda x d\lambda, \quad \tau_{zy} = -\mu \int_0^\infty \lambda A(\lambda) e^{-\lambda y} \cos \lambda x d\lambda$$

Satisfying the boundary conditions (1.30), we obtain after integrating τ_{zy} with respect to x

$$\int_0^\infty A(\lambda) \sin \lambda x d\lambda = \frac{2M}{\pi\mu}\left(x - \frac{x^2}{2}\right) \qquad 0 \leqslant x < 1 - a$$

$$\int_0^\infty A(\lambda) \cos \lambda x d\lambda = 0, \qquad x > 1 - a$$

Let us introduce a new unknown function $L(t)$

$$A(\lambda) = \int_0^{1-a} L(t)\, J_0(\lambda t)\, dt$$

Then

$$\int_0^x \frac{L(t)\, dt}{\sqrt{x^2 - t^2}} = \frac{2M}{\pi\mu}\left(x - \frac{x^2}{2}\right), \quad L(t) = \frac{2M}{\pi\mu}\left(t - \frac{2}{\pi}t^2\right)$$

$$\tau_{zy}(x, 0) = -\mu \frac{\partial}{\partial x}\int_0^\infty A(\lambda)\sin\lambda x d\lambda = -\mu\frac{\partial}{\partial x}\int_0^{1-a} L(t)\left[\int_0^\infty J_0(\lambda t)\sin\lambda x d\lambda\right] dt =$$

$$-\frac{2M}{\pi}(1 - x) - \delta\mu\frac{\partial}{\partial x}\int_0^{1-a}\frac{L(t)\, dt}{\sqrt{x^2 - t^2}}, \quad \delta = \begin{cases} 0, & 0 \leqslant x < 1 - a \\ 1, & x > 1 - a \end{cases} \quad (1.31)$$

Taking account of (1.31), the expression for the shear stresses on a continuation of the slit is the following:

$$\tau_{zy}(x, 0) = \frac{2Mx}{\pi\sqrt{x^2 (1-a)^2}}\left[1 - \frac{2(1-a)}{\pi}\right] + \ldots, \quad x > 1 - a \quad (1.32)$$

Terms bounded as $x \to 1 - a$ are discarded here. Taking account of (1.32), the stress intensity coefficient [2] is

$$K_{\mathrm{III}} = \lim_{x\to 1-a}\sqrt{2\pi[x - (1-a)]}\,\tau_{zy}(x, 0) = 2M\sqrt{\frac{1-a}{\pi}}\left[1 - \frac{2(1-a)}{\pi}\right] \quad (1.33)$$

Hence, in the limit case

$$M^* = \frac{3M}{4\sqrt{\pi}K_{\mathrm{III}C}} = \frac{3}{8\sqrt{1-\alpha}\,[1 - 2\pi^{-1}(1-\alpha)]}, \quad \alpha = \frac{a}{R} \quad (1.34)$$

The dash-dot line in Fig. 2 corresponds to (1.34) (R is the characteristic length introduced instead of unit length).

Combining all three solutions, the exact solution (1.27), the solution for the case of a deep slit (1.29), and the solution for a shallow groove (1.33), let us represent the stress intensity coefficient as follows ($\tau_{\max}$ is the maximum stress in a net-section):

$$K_{\mathrm{III}} = \tau_{\max}\sqrt{\pi R}F(\alpha), \quad \tau_{\max} = \frac{2M}{\pi a^3}, \quad \alpha = \frac{a}{R} \quad (1.35)$$

$$F(\alpha) = {}^3/_8\alpha^3(M^*)^{-1} \quad \text{(exact solution)}$$

$$F(\alpha) = F_N(\alpha) = {}^3/_8\alpha^{1/2} \quad \text{(deep slit)}$$

$$F(\alpha) = F_A(\alpha) = \alpha^3(1-\alpha)^{1/2}\left[1 - \frac{2}{\pi}(1-\alpha)\right] \quad \text{(shallow groove)}$$

Values of $F(\alpha)$, $F_N(\alpha)$ and $F_A(\alpha)$ are as follows:

α		0.1	0.3	0.5	0.6	0.7
$F(\alpha)$	10^3	0.119	0.206	0.264	0.288	0.286
$F_N(\alpha)$	10^3	0.118	0.205	0.265	0.290	0.313
$F_A(\alpha)$	10^3	—	—	—	—	—

α		0.8	0.85	0.9	0.95	1
$F(\alpha)$	10^3	0.274	0.243	0.231	0.210	0
$F_N(\alpha)$	10^3	0.336	—	—	—	—
$F_A(\alpha)$	10^3	—	0.221	0.218	0.207	0

2. Let us examine the case of axial tension by a force $P = q\pi R^2$ on a cylinder of unit radius with an annular slit (Fig. 1). Let us find the approximate solution of this problem under the assumption that the surface of the slit is free of traction, and the shear stresses and radial displacements are zero on the side surface of the cylinder. The problem is axisymmetric, and the state of stress in the neighborhood of the slit can be obtained from an analysis of a semi-infinite cylinder $z \geqslant 0$ for which the following conditions

$$\tau_{rz}(r,\ 0), \quad 0 \leqslant r < a; \quad | u_z(r,0) = 0, \quad 0 \leqslant r < a \qquad (2.1)$$
$$\sigma_z(r,\ 0) = -q, \quad a < r \leqslant 1$$

are satisfied on the endface $z = 0$. In this case the displacement and stress components can be expressed in terms ot one harmonic function [4]

$$u_r = z\frac{\partial^2\psi}{\partial r\,\partial z} + (1 - 2\nu)\frac{\partial\psi}{\partial r}, \qquad u_z = z\frac{\partial^2\psi}{\partial z^2} - 2(1-\nu)\frac{\partial\psi}{\partial z}$$
$$\sigma_z = 2\mu\left(z\frac{\partial^3\psi}{\partial z^3} - \frac{\partial^2\psi}{\partial z^2}\right), \qquad \tau_{zr} = 2\mu z\frac{\partial^3\psi}{\partial r\,\partial z} \qquad (2.2)$$

Taking account of the conditions as $z \to \infty$, we select the harmonic axisymmetric function in the form

$$\psi(r,\ z) = \sum_{n=1}^{\infty} \lambda_n^{-2} A_n J_0(\lambda_n r)\exp(-\lambda_n z) \qquad (2.3)$$

Here $J_0(x)$ is a first order Bessel's function, and λ_n are the roots of the equation $J_0'(\lambda_n) = 0$. The following conditions

$$\tau_{rz}(1,\ z) = 0, \qquad u_r(1,\ z) = 0 \qquad (2.4)$$

must be satisfied on the side surface of the cylinder. Satisfying conditions (2.1), we obtain the dual series equations

$$\sum_{n=1}^{\infty}\lambda_n^{-1} A_n J_0(\lambda_n r) = 0, \quad 0 \leqslant r < a$$
$$\sum_{n=1}^{\infty} A_n J_0(\lambda_n r) = \frac{q}{2\mu}, \quad a < r \leqslant 1 \qquad (2.5)$$

To solve this system, let us assume [1]

$$-\frac{q}{2\mu} + \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) = -\frac{1}{r}\frac{\partial}{\partial r}\int_r^a \frac{tg(t)\,dt}{\sqrt{t^2 - r^2}}, \qquad 0 \leqslant r < a \qquad (2.6)$$

It follows from the second equation of (2.5) and from (2.6)

$$A_n = 2J_0^{-2}(\lambda_n)\int_0^a g(t)\cos(\lambda_n t)\,dt \qquad (2.7)$$

$$\int_0^a g(t)\,dt = -\frac{q}{4\mu} \qquad (2.8)$$

Substituting (2.7) for the coefficients A_n in the first equation of (2.5), and taking account of the dependence obtained in [1], we find

$$g(t)=\int_0^a g(u)K(u,t)\,du+\frac{4}{\pi}\int_0^a g(u)\,du \tag{2.9}$$

$$K(u,t)=\frac{4}{\pi^2}\int_0^\infty \frac{K_1(y)}{yI_1(y)}\,[2I_1(y)-y\,\mathrm{ch}(uy)\,\mathrm{ch}(ty)]\,dy \tag{2.10}$$

Thus, the equality (2.9) is a Fredholm integral equation of the second kind to determine the function $g(t)$. To solve this equation, let us use a power series representation of the kernel (2.10) [5]:

$$K(u,t)=\sum_{m=0}^{\infty} b_{2m}(u)\,t^{2m} \tag{2.11}$$

Here

$$b_0(u)=\frac{4}{\pi^2}\left[T^*-\sum_{S=1}^{\infty} T_{2S}\frac{u^{2S}}{(2S)!}\right],\quad b_{2m}(u)=-\frac{4}{\pi^2}\sum_{S=1}^{\infty}\frac{T_{2m+2S-2}U^{2S-2}}{(2m)!\,(2S-2)!}\quad (m=1,2,\ldots)$$

$$T^*=\sum_{S=1}^{\infty}\frac{T_{2S}}{2^{2S}\,S!\,(S+1)!},\qquad T_n=\int_0^\infty \frac{K_1(y)}{I_1(y)}\,y^n dy \tag{2.12}$$

and the numerical values of T_n are presented in [5].

Expanding the solution of (2.9) in the series

$$g(t)=C(a)\sum_{m=0}^{\infty} Q_{2m}t^{2m} \tag{2.13}$$

we obtain an infinite system of algebraic equations to determine the coefficients Q_{2m}

$$Q_{2m}=\sum_{k=0}^{\infty} Q_{2k}C_{2k,2m}+\delta_m{}^{\circ},\qquad m=0,1,2,\ldots \tag{2.14}$$

Here

$$C_{2k,0}=\frac{4}{\pi^2}T^*\frac{a^{2k+2}}{2k+1}-\frac{4}{\pi^2}\sum_{S=1}^{\infty}\frac{T_{2S}a^{2k+2S+1}}{(2S)!\,(2S+2k+1)},\quad k=0,1,2,\ldots$$

$$C_{2k,2m}=-\frac{4}{\pi^2}\sum_{S=1}^{\infty}\frac{T_{2m+2S-2}a^{2k+2S-1}}{(2m)!\,(2S-2)!\,(2k+2S-1)},\quad k=0,1,2,\ldots,\ m=1,2,3,\ldots$$

The system (2.14) is quasi-regular for $0\leqslant a<1$. This follows from the asymptotic expressions for the functions $b_{2m}(u)$ for large m

$$|b_{2m}(u)|\sim\frac{2}{\pi(2-u)^{2m+1}}\left[1+\frac{3}{4}\,\frac{2-u}{2m}+\frac{9}{32}\,\frac{(2-u)^2}{(2m-1)\,2m}+\cdots\right]+\frac{2}{\pi(2+u)^{2m+1}}\left[1+\frac{3}{4}\,\frac{2+u}{2m}+\frac{9}{32}\,\frac{(2+u)^2}{(2m-1)\,2m}+\cdots\right],\quad 0\leqslant u<1$$

and the estimates

$$|C_{2k,2m}|\leqslant|b_{2m}(a_1)|\,\frac{a^{2k+1}}{2k+1},\qquad 0\leqslant a_1<a<1$$

$$S_{2m+1}=\sum_{k=0}^{\infty}|C_{2k,2m}|\leqslant\frac{1}{2}|b_{2m}(a_1)|\ln\frac{1+a}{1-a}$$

Thus, starting with some number $m = m'$ the following inequality is valid

$$S_{2m+1} < 1, \qquad m \geqslant m'$$

Let us indicate here an asymptotic formula to determine the coefficients T_n for large n

$$T_n \sim \frac{\pi n!}{2^{n+1}}\left[1 + \frac{3}{4}\,\frac{2}{n} + \frac{9}{32}\,\frac{2^2}{(n-1)\,n} + \cdots\right]$$

For $n \geqslant 8$ the values of T_n calculated by means of this formula differ by less than 1.1% from the exact values. We define the constant $C(a)$ so that condition (2.8) would be satisfied. We consequently obtain

$$C(a) = -\frac{q}{\pi G}\left(\sum_{m=0}^{\infty} Q_{2m}\frac{a^{2m+1}}{2m+1}\right)^{-1} \tag{2.15}$$

The normal stresses in the $z = 0$ plane on the continuation of the slit will be determined from the formula

$$\sigma_z(r, 0) = -2\mu\sum_{n=1}^{\infty} A_n J_0(\lambda_n r) = -q + \frac{2\mu}{r}\frac{\partial}{\partial r}\int_r^a \frac{tg(t)\,dt}{\sqrt{t^2 - r^2}}, \qquad r \leqslant a \tag{2.16}$$

We substitute (2.13) into (2.16), and isolate the singularity for the stresses at the vertex of the slit. We then obtain

$$\sigma_z(r, 0) = -\frac{2GC(a)}{\sqrt{a^2 - r^2}}\sum_{m=0}^{\infty} Q_{2m} r^{2m} + \cdots, \qquad r < a \tag{2.17}$$

The terms bounded as $r \to a$ are discarded here. Formula (2.17) permits determination of the stress intensity coefficient [2] and the quantity Q_I proportional to the critical load

$$K_{\rm I} = \sqrt{2\pi}\lim_{r\to a}\left[\sqrt{a - r}\,\sigma_z(r, 0)\right] = -2G\sqrt{\frac{\pi}{a}}\,g(a), \qquad r < a \tag{2.18}$$

$$Q_{\rm I} = 2qR^{1/2}\pi^{-1/2}K_{\rm Ic}^{-1}$$

Taking account of (2.15) for the constant $C(a)$, the stress intensity coefficient in the case of a cylinder of radius R is

$$K_{\rm I} = \frac{q\sqrt{\pi R}}{2\alpha^{3/2}}\left(\sum_{m=0}^{\infty} Q_{2m}\alpha^{2m}\right)\left(\sum_{m=0}^{\infty} Q_{2m}\frac{\alpha^{2m}}{2m+1}\right)^{-1}, \qquad \alpha = \frac{a}{R} \tag{2.19}$$

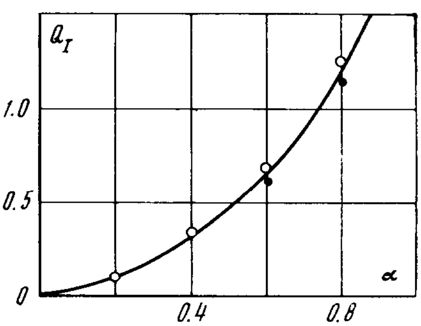

Fig. 4

For small values of α we have $\theta_0 \sim 1$, $\theta_{2m} \sim 0$ $(m = 1, 2, \ldots)$ and it follows from (2.19) that

$$K_{\rm I} \sim {}^1/_2 q\sqrt{\pi R}\,\alpha^{-3/2}$$

This result derives from the Neuber [3] solution of the problem on tension of an unbounded solid with an external slot.

The dependence of the quality $Q_{\rm I}$ on the distance to the vertex of the slit $\alpha = a / R$ is shown in Fig. 4. Presented here for comparison are the Paris (open circles) and Bückner (dark circles) data [6].

It is obvious that the result obtained above occupies an intermediate position.

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Translated by M. D. F.

UDC 539.375

A SYSTEM OF ARBITRARILY ORIENTED CRACKS IN ELASTIC SOLIDS

PMM Vol. 37, №2, 1973, pp. 326-332

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(Received April 21, 1972)

The plane problem of the theory of elasticity for an unbounded domain, containing N arbitrarily situated rectilinear cuts (cracks), is reduced to a system of N singular integral equations relative to functions which characterize the discontinuity of the displacements along the crack lines. The general solution of the integral equations for the case of distantly located cracks in the form of a power series with respect to a small parameter, is obtained. The problem of rupture is also considered.

In the plane theory of cracks there exist a series of investigations devoted to the study of the interactions between cracks which are ordered in a definite manner (colinear [1−3], parallel [4, 5], with a chessboard distribution [6]). By the representation of the complex potentials in the form of Laurent series [7], we determine approximately the state of stress of an unbounded plate, weakened by a system of arbitrarily oriented cracks, in the case of a linear distribution of stresses at infinity. We reduce the plane problem of the theory of elasticity for an infinite body, containing arbitrarily situated rectilinear cracks and with an arbitrary load, to a system of integral equations; this will allow to solve a series of new problems in the mathematical theory of cracks.

1. Assume that in an elastic plane, related to a Cartesian system of coordinates xOy, there exist N cuts (cracks) of length $2a_k$ $(k = 1, 2, \ldots, N)$. The centers O_k of the cracks are determined by the coordinates $z_{k0} = x_{k0} + iy_{k0} = d_k e^{i\beta_k}$. At the points O_k there are located the origins of local systems of coordinates $x_k O_k y_k$. The axes $O_k x_k$ coincide with the crack lines and form the angles α_k with the axis Ox (Fig. 1). The boundaries of the cracks are loaded by the self-balancing forces